Optical Conductivity and Pseudo-Momentum Conservation in Anisotropic Fermi Liquids*

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Umklapp scattering determines the conductivity of clean metals. In typical quasi one-dimensional Fermi liquids with an open Fermi surface, certain pseudo-momenta do not decay by 2-particle collisions even in situations where Umklapp scattering relaxes the momentum of the quasi particles efficiently. Due to this approximate conservation of pseudo-momentum, a certain fraction of the electrical current decays very slowly and a well-pronounced low-frequency peak emerges in the optical conductivity. We develop simple criteria to determine under what conditions approximate pseudo-momentum conservation is relevant and calculate within in Fermi liquid theory the weights of the corresponding low-frequency peaks and the temperature dependence of the various relevant decay rates. Based on these considerations, we obtain a qualitative picture of the frequency and temperature dependence of the optical conductivity of an anisotropic Fermi liquid.

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1. INTRODUCTION

The transport properties of clean metals at finite temperatures are determined by Umklapp scatterings which allow the electrical current to decay by transfering momentum to the underlying crystal structure in quantas given by the reciprocal lattice vectors $\mathbf{G}_i$ [1,2] . The decay rates are small for a clean metal with a small Fermi surface and accordingly the conductivities can be very large. In the case of a large Fermi surface, momentum can relax efficiently by Umklapp scattering. Quasi one-dimensional metals

*dedicated to Peter Wölfle on the occasion of his 60th birthday

with an open Fermi surface are a particularly interesting case, as certain pseudo-momenta are still approximately conserved impeding the current decay "protecting" a certain fraction of the current[3]. This gives rise to well pronounced low-frequency peaks in the optical conductivity as we will explain in detail below.

In this paper we shall introduce these approximately conserved "pseudo-momenta" and demonstrate that multi-particle processes are required to violate them. After a general discussion of the optical conductivity $\sigma(\omega)$ in the presence of the approximately conserved quantities, we will investigate in detail under what conditions the decay of certain pseudo-momenta is strongly suppressed. Finally, we study how these pseudo-momenta influence the optical conductivity and the temperature dependence of the resitivity.

2. APPROXIMATE CONSERVATION LAWS AND OPTICAL CONDUCTIVITY

We wish to study how weakly violated conservation laws and the resulting slowly decaying modes influence the optical conductivity. Consider a situation where a quantity $\tilde{P}$ exists (to be refered to as "pseudo-momentum") with the following two properties: i) $\tilde{P}$ is approximately conserved ii) the current J has a finite cross-susceptibility $\chi_{J\tilde{P}}$ with the pseudo-momentum.

We assume that the pseudo-momentum is approximately conserved in the sense that its decay rate is slower than any other quantity Q with $\chi_{Q\tilde{P}} = 0$ ("perpendicular" to $\tilde{P}$) and $\chi_{QJ} \neq 0$. In the following sections we will give a number of examples of such pseudo-momenta, where the commutator $[H, \tilde{P}] \approx 0$ vanishes for most of the relevant low-energy processes. In particular we shall show that anisotropic Fermi Liquids possess such approximate conservation laws. The second requirement implies that a typical state with a finite expectation value of the current will also carry a finite pseudo-momentum and similarly a state with finite pseudo-momentum will also be characterized by a finite current: J and $\tilde{P}$ have a finite "overlap".

Now proceed with the following Gedanken experiment (see Fig. 1): prepare a state with a finite current $\langle J(t=0)\rangle > 0$ and switch off the driving electric field at time $t = 0$. As the current is not conserved, it will decay rather fast with a typical rate which we denote by Γ_J. The initial state with finite current will also have a finite pseudo momentum $\langle \tilde{P}(t=0)\rangle$ with $\frac{\langle \tilde{P}(t=0)\rangle}{\langle J(t=0)\rangle} = \frac{\chi_{\tilde{P}J}}{\chi_{JJ}}$ which will decay at a much slower rate than the current; $\Gamma_{\tilde{P}} \ll \Gamma_J$. The slow decay of the pseudo-momentum will, as time goes on, induce a slow decay of the current since a state with finite pseudo-momentum will typically carry a finite current $\langle J(t)\rangle = J(\langle \tilde{P}\rangle) \approx \frac{\chi_{J\tilde{P}}}{\chi_{\tilde{P}\tilde{P}}} \langle \tilde{P}(t)\rangle$. Thus, in

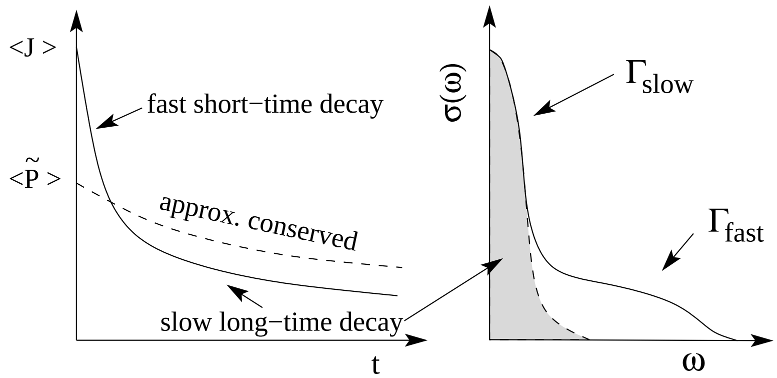

Fig. 1. In a Gedanken experiment, a state with current $\langle J\rangle$ is prepared. At time $t=0$, the external electric field is switched off. J decays rapidly while the pseudo-momentum $\tilde{P}$ is approximately conserved and has a much lower decay rate $\Gamma_{\tilde{P}}$. In the optical conductivity $\sigma(\omega)$ the approximate conservation of $\tilde{P}$ leads to a low-frequency peak with the width $\Gamma_{\tilde{P}}$ and a height proportional to $1/\Gamma_{\tilde{P}}$. In an anisotropic Fermi liquid, the width $\Gamma_{\tilde{P}}$ is given by Eqn. (10), the weight by (1).

the long time limit, a finite fraction,

$$\frac{D}{D_0}=\frac{\chi^2_{J\tilde{P}}}{\chi_{\tilde{P}\tilde{P}}\chi_{JJ}}, \qquad (1)$$

of $\langle J\rangle$ will not decay with the fast rate Γ_J but with the much smaller rate $\Gamma_{\tilde{P}}$ as is shown schematically in Fig. 1.

How does this affect the optical conductivity? We argue (supported by rigorous arguments[4], analytical calculations[3] and numerical simulations[10], see below) that a slow long-time decay of some fraction of the current leads to a corresponding long-time tail in the (equilibrium) current-current correlation function from which the optical conductivity is calculated. Therefore, one expects a low-frequency peak in the optical conductivity which carries the fraction D/D_0 of the total weight[6] $\pi\chi_{JJ}=2\pi D_0=\pi\frac{ne^2}{m}$ with $\frac{n}{m}=\sum_{\mathbf{k}\sigma}\frac{\partial^2\epsilon_k}{\partial k^2}\langle c^\dagger_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma}\rangle$.

The width of the low-frequency peak is determined by the decay-rate of $\tilde{P}$. As the weight of a peak is approximately given by its width multiplied with its height, we expect that the height and therefore the dc-conductivity is of the order of $D/\Gamma_{\tilde{P}}$. The peak is well pronounced if its height is well above the “background”, i.e. if

$$\frac{D}{\Gamma_{\tilde{P}}}\gg\frac{D_0}{\Gamma_J}. \qquad (2)$$

In this case, the dc-conductivity is determined by the decay rate $\Gamma_{\tilde{P}}$ of $\tilde{P}$.

A particular case is the limit $\Gamma_{\tilde{P}} \to 0$, i.e. when $\tilde{P}$ is *exactly conserved*. In this situation, studied long time ago in a more general contex by Mazur[4] and Suzuki[5], the hand-waving arguments given above can be made rigorous. The low-frequency peak evolves into a true δ-function and the optical conductivity takes the form $\sigma(\omega, T) = 2\pi D(T)\delta(\omega) + \sigma_{\text{reg}}(\omega, T)$ with a finite Drude weight D. For a system with conserved charges $Q_n, n = 1 \dots M$, with $\chi_{Q_n Q_m} = 0$ for $n \neq m$, the Mazur inequality reads,

$$D \geq \frac{1}{2} \sum_{n=1}^{M} \frac{\chi_{JQ_n}^2}{\chi_{Q_n Q_n}} \tag{3}$$

Furthermore, Suzuki[5] showed that the inequality in (3) can be re placed by an equality if the sum includes *all* conservation laws! If therefore $\tilde{P}$ is the *only* (approximately) conserved quantity in the system with a finite overlap to the current $\chi_{JQ} \neq 0$ (as assumed above) then (1) is exact as was tested numerically for a simple model in Ref. 10. The importance of the inequality for transport has been recently emphasized by Zotos *et al.*[7].

The above discussed low frequency peak in $\sigma(\omega)$ with weight D should not be confused with the zero-temperature Drude weight $D(T = 0)$ as in general, $\lim_{T \to 0} D(T)$ need *not* be identical to $D(T = 0)$. At low, but finite temperature, the low frequency peak is well defined as $\Gamma_{\tilde{P}} \ll \Gamma_J \frac{D}{D_0}$. At $T = 0$ in a clean metal, one expects $\Gamma_{\tilde{P}} = \Gamma_J = 0$ and therefore it is not possible to extract the weight of the peak which is due to the slow mode $\tilde{P}$. The same situation arises in finite temperature experiments when the temperature is so low that Γ_J is smaller than the energy resolution.

3. TRANSPORT IN AN ANISOTROPIC FERMI LIQUID

3.1. Pseudo-Momentum

We introduce in this section pseudo-momentum operators conserved by generic 2-particle low energy scattering terms. Since multi-particle or high-energy processes are required to violate their conservation laws the associated decay times can be very long at low T.

We consider a three- or two-dimensional anisotropic metal with a clearly defined most-conducting axis in x-direction. It is assumed that two well defined Fermi sheets perpendicular to this axis exist (see Fig. 2). The curvature of those sheets is not required to be very small (see below). For simplicity, we discuss only situations where a single band crosses the Fermi energy, but many of our results can be generalized to multi-band models. We consider

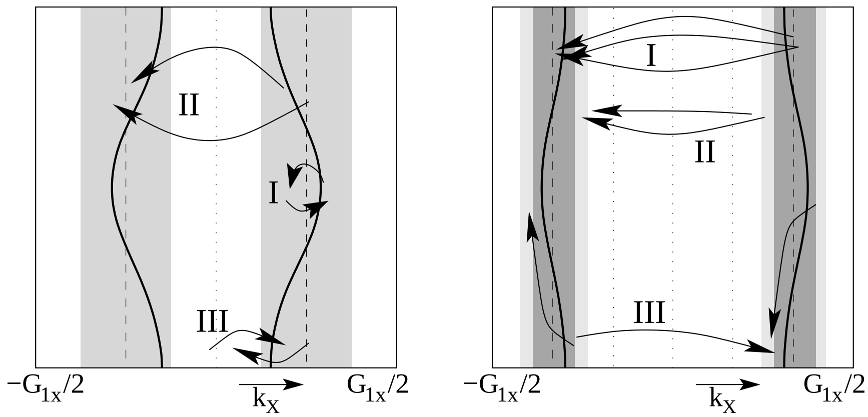

Fig. 2. Left figure: Fermi surface (FS) of an anisotropic metal close to half filling. Both "forward" (I) and "Umklapp" (II) scattering processes do not lead to an decay of the pseudo-momentum $\tilde{P}_{21}$ as long as the momenta are within the shaded area. The scattering event III leads to a decay of $\tilde{P}_{21}$ but is exponentially suppressed at low T as it involves quasi particles far away from the Fermi surface. Right figure: Filling close to 2/3. Process I picks up the momentum $2\mathbf{G}_1$ and relaxes both P_x and $\tilde{P}_{21}$ but not $\tilde{P}_{32}$. The 2-particle scattering events of type II change $\tilde{P}_{32}$ but are exponentially suppressed at low T as long as the FS is in the shaded region. The 3-particle process III is only suppressed for a FS within the dark-shaded region (see Eqn. (7)).

a rather arbitrary lattice, assuming only the existence of a translation vector $\mathbf{a}_1$ of the underlying lattice in the x-direction, $\mathbf{G}_1$ is the corresponding reciprocal vector with $\mathbf{a}_1\mathbf{G}_1 = 2\pi$

Umklapp processes lead to a decay of any macroscopic momentum. For example, close to half filling, the process II shown in Fig. 2 relaxes the momentum in x direction via a momentum transfer G_1^x. More generally, in the case of a filling fraction close to M_0/N_0 with integers M_0 and N_0, the Fermi momentum is approximately given by $k_F^x \approx \frac{M_0}{N_0}\frac{G_x^1}{2}$ and therefore Umklapp processes where N_0 particles move from one Fermi sheet to the other while picking up a lattice momentum $M_0 G_x^1$ dominate the momentum relaxation for sufficiently flat Fermi surfaces (see below).

We define now a pseudo-momentum operator $\tilde{P}_{N_0 M_0}$ with the property that it is conserved by two-particle scattering processes close to the FS. It will decay only via high energy two-particle or multi-particle processes, and as a result has a very slow decay. We term such quantities "approximately conserved". The pseudo-momentum is obtained by measuring the momentum in x-direction on the left/right Fermi sheet with respect to the line $k_x = \pm\frac{M_0}{N_0} G_x^1$ (dashed line in Fig. 2),

$$\tilde{P}_{N_0 M_0} = P_x - \frac{M_0}{N_0}\frac{G_x^1}{2}(N_R - N_L) = \sum_{\mathbf{k}} \delta k_{N_0 M_0} c^\dagger_{\mathbf{k}} c_{\mathbf{k}} \tag{4}$$

where $N_{R/L} = \sum_{k_x \gtrless 0} c^\dagger_{\mathbf{k}} c_{\mathbf{k}}$ is the number of "right-moving" (left-moving) electrons and,

$$\delta k_{N_0 M_0} = k_x - \frac{M_0}{N_0}\frac{G_x^1}{2}\operatorname{sgn}(k_x). \tag{5}$$

To check to what extent $\tilde{P}_{N_0 M_0}$ is conserved, consider a generic 2-particle scattering term:

$$H_2 = \sum_{1.\ \mathrm{BZ}} c^\dagger_{\mathbf{k}_1} c^\dagger_{\mathbf{k}_2} c_{\mathbf{q}_2} c_{\mathbf{q}_1} V_{\mathbf{k}_1\mathbf{k}_2,\mathbf{q}_1\mathbf{q}_2} \sum_{\mathbf{G_n}} \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{q}_1 - \mathbf{q}_2 - \mathbf{G_n})$$

and calculate the commutator of $\tilde{P}$ with it. We find,

$$\begin{aligned}[\tilde{P}_{N_0 M_0}, H_2] &= \sum_{1.\ \mathrm{BZ}} c^\dagger_{\mathbf{k}_1} c^\dagger_{\mathbf{k}_2} c_{\mathbf{q}_2} c_{\mathbf{q}_1} V_{\mathbf{k}_1\mathbf{k}_2,\mathbf{q}_1\mathbf{q}_2} \sum_{\mathbf{G_n}} \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{q}_1 - \mathbf{q}_2 - \mathbf{G_n}) \\ &\quad \times(\delta k^1_{N_0 M_0} + \delta k^2_{N_0 M_0} - \delta q^1_{N_0 M_0} - \delta q^2_{N_0 M_0})\end{aligned} \tag{6}$$

where the $\mathbf{G}_n$ are reciprocal lattice vectors. For half filling, the right-hand side of (6) vanish if all four momenta are in the shaded region of Fig, 2, i.e. $|\delta k^1_{21}|$, $|\delta k^2_{21}|$, $|\delta q^1_{21}|$, $|\delta q^2_{21}| < G_{1x}/4$: For "forward scattering" processes of type I in Fig, 2, $k_{1x} + k_{2x} - k'_{1x} - k'_{2x} = 0$, both momentum P_x and pseudo momentum $\tilde{P}$ are conserved as for all 4 momenta have the same sign. "Umklapp" processes of type II pick up a lattice momentum G_x which is exactly compensated as two electrons are moved from the right to the left Fermi surface due to the term $\operatorname{sign}[k_x]G_{1x}/4$ in the definition of δk_{21} (5), and $\tilde{P}$ is again conserved. The pseudo-momentum can only decay by two-particle high-energy processes far from the Fermi surface, e.g. III in Fig. 2. Such scattering processes is exponentially suppressed even at moderate temperatures. Note, that the usual version of the Boltzmann equation, only 2-particle scattering processes are taken into account.

Low-energy contributions can, however, result from these high-energy processes due to virtual excitations (in high orders of perturbation theory). Or to put it in the language of renormalization group: N-particle interactions are generated. Will they relax $\tilde{P}_{N_0 M_0}$? Due to conservation of lattice momentum, P_x can change only in quanta of G_x^1 (note that other reciprocal lattice vectors are perpendicular to $\mathbf{a}_1$) and particle conservation guarantees that a change of $N_R - N_L$ will occur in multiples of 2. Therefore $\tilde{P}_{N_0 M_0}$ can be

altered only by an amount $G_x^1\left(\Delta m + \Delta n \frac{M_0}{N_0}\right)$ with integers Δm and Δn, i.e. the smallest possible change of the pseudo-momentum is $\Delta \tilde{P}_{N_0 M_0} = \frac{1}{N_0} G_x^1$. Therefore, a relaxation of $\tilde{P}_{N_0 M_0}$ is not possible if all $2N$ pseudo-momenta δk_{ix} involved in an N-particle scattering process are smaller than $\frac{1}{2N}\frac{1}{N_0} G_x^1$. For a given Fermi surface, the decay of $\tilde{P}$ by N-particle collision at low T is possible only for

$$N > N^*_{N_0 M_0} = \frac{G_{1x}/(2N_0)}{\max |\delta k^F_{N_0 M_0}|}, \tag{7}$$

where $\max |\delta k^F_{N_0 M_0}|$ is the maximal distance of the Fermi surface from the plane $k_x = \pm \frac{M_0}{2N_0} G_{1x}$ (dashed line in Fig. 2). This simple geometrical criterion together with its consequences discussed below, is the central result of this paper. If one sets $N_0 = 1$ and $M_0 = 0$ or 1, one obtains a criterion for the decay of the momentum P_x. Note that (7) is a necessary condition, there can be situations, where it is not sufficient.

At sufficiently high temperatures, the broadening of the Fermi-surface and the thermal excitation of states with higher energy will favor decay channels of the pseudo-momentum with smaller N (this effect can crudely be described by adding T/v_F to $\max |\delta k^F_{N_0 M_0}|$ in (7)).

We can at this point define precisely what is meant by the terms "close to a commensurate filling" or "small curvature of the Fermi surface" in the context of this paper. Given a Fermi surface and a filling we wish to find which pseudo-momentum $\tilde{P}_{\tilde{N}_0 \tilde{M}_0}$ will have the longest decay times at low T. As processes with large N are suppressed at low temperature due to phase-space restrictions (see below), the answer can be obtained by determining the natural numbers $\tilde{N}_0$ and $\tilde{M}_0$ for which N^* is maximal,

$$N^*_{\tilde{N}_0 \tilde{M}_0} = \max_{N_0, M_0} N^*_{N_0 M_0}. \tag{8}$$

In the right panel of Fig. 2 the various relevant scattering processes are discussed in a situation where $\tilde{P}_{32}$ is the most slowly decaying pseudo-momentum in the system. If the Fermi surface is within the dark-shaded area (defined by setting $N = 3$ on the left-hand side of Eqn. 7), then $\tilde{P}_{32}$ will decay neither by two- nor by three-particle scattering events.

In general, approximate conservation laws will be important for any clean system at low temperatures if the relevant momentum or pseudo-momentum *cannot* decay by the usual 2-particle processes, i.e. if

$$N^*_{\tilde{N}_0 \tilde{M}_0} > 2. \tag{9}$$

For a generic 3-dimensional Fermi surface, $N^*_{\tilde{N}_0 \tilde{M}_0}$ will typically be smaller than 2: two-particle Umklapp processes efficiently lead to a decay of the

current and all relevant momenta and pseudo-momenta. The situation is, however, different for quasi one-dimensional metals with two well defined Fermi sheets with moderate curvature and also for systems with a small FS. For example, for $N_0 = 2$, it suffices if the FS is within the shaded region of Fig. 2 as already discussed above.

If $\tilde{N}_0 = 1$, then the usual momentum P_x is the most important approximate conservation law (with $\tilde{M}_0 = 0$ for a particle-like Fermi surface and $\tilde{M}_0 = 1$ if a hole picture is more appropriate). This will be the case for a small density of particles (or holes) and more generally for arbitrary filling if the Fermi surface is closed. For an open Fermi surface, the factor $1/N_0$ in (7) guarantees that for moderate curvature of the Fermi surface only small values of N_0 are relevant. Large values of $\tilde{N}_0$ can arise for incommensurate fillings in the extreme quasi one-dimensional limit, when $\delta k^F_{N_0 M_0}$ is very small.

3.2. Decay Rates and dc-Conductivity

The temperature dependence of the decay-rate $\Gamma_{\tilde{P}_{N_0 M_0}}$ of $\tilde{P}_{N_0 M_0}$ at low T in the Fermi liquid regime is determined by the usual phase-space arguments: a particle of energy $\omega \sim T$ decays into $2N^* - 1$ particle and hole excitations, one of the energies is fixed by energy conservation, and the remaining $2N^* - 2$ energies each have a phase-space of order ω. Therefore,

$$\Gamma_{\tilde{P}_{N_0 M_0}} \propto T^{2N-2} \tag{10}$$

where the integer N is the smallest value consistent with (7). The prefactor in (10) depends in a rather delicate way on the strength and range of the interaction, the screening and the band-curvature. Note, that a *local* N-particle interaction will give no contribution for $N > 2$ due to the Pauli-principle. Therefore the scattering rate is strongly suppressed for weakly coupled chains with well-screened interactions. Furthermore, additional logarithmic temperature dependences of the scattering vertices are expected even in the Fermi-liquid regime as it is well known from Fermi liquid theory.

We want to stress that the analysis given above is valid for interactions of arbitrary strength as long as a Fermi liquid description is possible. For strong interaction, one should, however, consider the pseudo-momentum of quasi particles which slightly differs from the pseudo-momentum of the bare electrons (see below).

In section 2., we have discussed how the decay rate of the pseudo-momentum determines the dc-conductivity. If $\chi^2_{J\tilde{P}_{\tilde{N}_0\tilde{M}_0}} / \chi_{\tilde{P}_{\tilde{N}_0\tilde{M}_0}\tilde{P}_{\tilde{N}_0\tilde{M}_0}}$ is finite for $T \to 0$ (which is the generic situation if the filling is not exactly

commensurate as we will discuss in the next two sections), then at lowest temperatures, the T-dependence of the dc-conductivity of a clean Fermi liquid will be determined by the decay rate of the pseudo-momentum and therefore

$$\sigma(\omega = 0, T) \propto T^{-(2N^*-2)} \tag{11}$$

where N^* is the smallest integer larger than $N^*_{\tilde{N}_0 \tilde{M}_0}$! This result is also valid in the case of a small Fermi surface[8], where the well-known momentum P_x is the most important conservation law. In this case, it differs from the often cited result by Peierls[1,2] $\sigma(T) \propto T^{-2} e^{E_0/T}$. The latter result is obtained e.g. within the usual Boltzmann equation, which takes into account only two-particle scattering processes that are gapped by the energy E_0 for a small FS and neglects higher order processes. In many realistic situations, even weak disorder or phonons will be the dominant relaxation mechanisms of the pseudo-momenta and (11) will not apply. We will argue below that, nevertheless, the approximate conservation of pseudo-momentum still remains experimentally relevant.

As both the optical and the dc conductivity are determined not only by decay rate of pseudo-momentum but also by the weight of the corresponding low-frequency peak, we will calculate the latter in the next two sections. Then, one can use Eqn. (2) to determine, whether pseudo-momentum conservation is relevant and under what conditions (11) applies.

3.3. Overlap of Current and Pseudo-Momentum: $\chi_{J_x \tilde{P}}$

To obtain the weight of the low-frequency peak in $\sigma(\omega, T)$, we have to calculate according to (3) or (1) the two susceptibilities $\chi_{J_x \tilde{P}_{N_0 M_0}}$ and $\chi_{\tilde{P}_{N_0 M_0} \tilde{P}_{N_0 M_0}}$, where J_x is the electrical current in x direction. In this subsection we consider the overlap $\chi_{J_x \tilde{P}_{N_0 M_0}}$.

This overlap is almost completely fixed by current conservation. Static susceptibility can be calculated by taking the limit $\omega \to 0$ first and then $\mathbf{q} \to 0$ (for the dc-conductivity the opposite limit is relevant). The continuity equation for charge for $q_y, q_z = 0, q_x \to 0$ reads

$$\frac{\partial}{\partial t} \rho_{q_x}(t) + i q_x J_x(t, q_x) = 0 \tag{12}$$

and the susceptibility is given by

$$\chi_{J_x \tilde{P}_{N_0 M_0}} = -i \lim_{q_x \to 0, \epsilon \to 0} \int_0^\infty dt e^{-\epsilon t} \left\langle \left[J_x(t, q_x), \tilde{P}_{N_0 M_0}(0, -q_x) \right] \right\rangle$$

$$
\begin{aligned}
&= \lim_{q_x \to 0} \frac{1}{q_x} \left\langle \left[\rho_{q_x}(t=0), \tilde{P}_{N_0 M_0}(t=0,-q_x) \right] \right\rangle \\
&= -e \sum_{\mathbf{k} \text{ in 1. BZ},\sigma} \delta k_{N_0 M_0} \frac{\partial}{\partial k_x} \left\langle c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} \right\rangle \\
&= e \Delta n_{N_0 M_0} + B_{\text{boundary}} \qquad (13)
\end{aligned}
$$

where ϵ is infinitesimally small and $\Delta n_{N_0 M_0} = n - \frac{M_0}{N_0} n_{\max}$, is the deviation of the particle density from the filling M_0/N_0 (in units of particles per volumem, $n_{\max}$ corresponds to the density of 2 electrons per unit cell). In the last step we used a partial integration which produced boundary terms at the center and the edge of the first Brillouin zone (BZ). For the square BZ shown in Fig. 2 they take the form

$$
B_{\text{boundary}} = e G_{1x} \Big(\sum_{\substack{\mathbf{k} \\ k_x = 0}} \frac{\left\langle c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} \right\rangle - 1}{2} + \sum_{\substack{\mathbf{k} \\ k_x = \pm G_{1x}/2}} \frac{\left\langle c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} \right\rangle}{4} \Big).
$$

The boundary terms are obviously very small for an open Fermi surface: deep in the Fermi sea, the occupation $\langle c^\dagger_{k_x=0} c_{k_x=0} \rangle$ is very close to 1, and at the edge of the Brillouin zone it is almost 0. This is not true for strong interactions, but as mentioned above, we redefine $\tilde{P}_{N_0 M_0}$ as the pseudo-momentum of the quasi particles. The probability to find a quasi particle (hole) close to the BZ boundary (at $k_x = 0$) is exponentially small for the systems discussed here. Note that Luttinger's theorem guarantees that $\Delta n_{N_0 M_0}$ for quasi particles is the same as for particles. Therefore we conjecture that the boundary terms are exponentially small,

$$
B_{\text{boundary}} \sim e^{-\beta \epsilon_F}, \qquad (14)
$$

if $\tilde{P}_{N_0 M_0}$ is appropriately defined. It is easy to convince oneself that this is true within Fermi liquid theory, we claim, however, that (14) and (13) hold under much more general conditions. The assumption, that boundary terms are exponentially small and therefore negligible is for example implicitly assumed in the standard derivation of the Luttinger liquid in one-dimensional systems (in the continuum description which is the basis of the Luttinger model, boundary contributions vanish exactly).

3.4. $\chi_{\tilde{P}_{N_0M_0}\tilde{P}_{N_0M_0}}$ and Low-Frequency Weight in Fermi Liquid Theory

The low-frequency weight D also requires $\chi_{\tilde{P}_{N_0M_0}\tilde{P}_{N_0M_0}}$ which can easily be calculated at low T within Fermi liquid theory following standard text books[9]. The result will in general depend on the details of the momentum dependence of the effective interactions and the band-structure. We assume a quasi 1D system with a Fermi velocity $v_F^* = k_F/m^*$ parallel to the most conducting axis and, for simplicity, completely local interactions characterized by two Fermi liquid parameter F_{++} and F_{+-} in the spin-singlet channel to describe the interactions of two density excitations $\delta n_{\mathbf{k}}$ on the same Fermi sheet or on different sheets, respectively. Then the relative weight (1) of the low frequency peak in the optical conductivity for low T is given by

$$\frac{D}{D_0} \approx \frac{m}{m^*}\left(\frac{\left\langle(\delta k_{N_0M_0})^2\right\rangle_{\mathrm{FS}}}{\langle\delta k_{N_0M_0}\rangle_{\mathrm{FS}}^2} - \frac{F_m}{1+F_m}\right)^{-1}. \tag{15}$$

$\langle\ldots\rangle_{\mathrm{FS}}$ is defined as an average over the Fermi sheet, e.g. $\langle\delta k_{N_0M_0}\rangle_{\mathrm{FS}} = \iint dk_y dk_z (k_F^x - \frac{M_0}{N_0}\frac{G_x}{2})/(\iint dk_y dk_z)$, where $k_F^x = k_F^x(k_y, k_z)$ is the x-component of the Fermi momentum on the right Fermi sheet, and $F_m = F_{++} - F_{+-}$. Note that due to Luttinger's theorem $\Delta n_{N_0M_0} = 2\langle\delta k_{N_0M_0}\rangle_{\mathrm{FS}}/(a_y a_z \pi)$, where $\Delta n_{N_0M_0}$ is the deviation of the electron-density from half filling.

If the interactions are sufficiently weak so that no phase transition is induced, the low-frequency weight D vanishes close to half filling with

$$\frac{D}{D_0} \sim \frac{m}{m^*}\left(\frac{\epsilon_F^*}{t_\perp^*}\right)^2\left(\frac{\Delta n_{N_0M_0}}{n}\right)^2, \tag{16}$$

where $\epsilon_F^* = k_F v_F^*$ is the renormalized Fermi energy. We expect that the low-frequency weight D decreases with increasing temperature, mainly due to the thermal broadening of $\left\langle(\delta k_x)^2\right\rangle_{\mathrm{FS}}$. Leading finite-$T$ corrections to (15) or (16) are of order $(T/\epsilon_F^*)^2$.

3.5. Qualitative Picture of the Optical Conductivity

Combining the results of the previous sections, one can obtain a qualitative picture of the low-temperature optical conductivity in a strongly correlated Fermi liquid. From the arguments given in this paper, very little can be said about $\sigma(\omega)$ at high frequencies of the order of the (renormalized) Fermi energy, the behavior will in general depend on details of the interactions and the band structure. In a quasi one-dimensional situation and

somewhat lower frequencies (still above the scale where a Fermi liquid description is possible) one might obtain results typical for a Luttinger liquid with Umklapp scattering[11,3] (in the Luttinger liquid regime, not discussed here, the conservation of pseudo-momenta can become important[3]). In the Fermi liquid regime, which is at the focus of this paper, two-particle collisions define the shortest relevant time-scale and therefore one expects the well-know "Drude" peak in the optical conductivity with a width $\Gamma \propto T^2$ given by the two-particle transport-rate, which we identify with the fast decay rate Γ_{J_x} of section 2. If some pseudo-momentum is approximately conserved and decays on a much longer time-scale at low temperatures, we expect a low-frequency peak in the optical conductivity (sketched in Fig. 1) on top of the much broader T^2 "Drude" peak. Whether pseudo-momentum conservation is important at lowest temperatures, depends on the structure of the Fermi surface. To find the most relevant $\tilde{P}_{N_0 M_0}$ in the limit $T \to 0$, we determine $N^*_{\tilde{N}_0 M_0}$ defined in Eqn. (7) and maximize it, to obtain the "best conserved" pseudo-momentum $\tilde{P}_{\tilde{N}_0 \tilde{M}_0}$, see Eqn. (8). If $\chi_{J\tilde{P}_{\tilde{N}_0 \tilde{M}_0}} \neq 0$, i.e. if the filling is *not* exactly $\tilde{M}_0/\tilde{N}_0$ (see Eqn. (13)), and if two-particle scattering processes do not relax $\tilde{P}_{\tilde{N}_0 \tilde{M}_0}$, i.e. $N^*_{\tilde{N}_0 \tilde{M}_0} > 2$, then a low-frequency peak is expected in the optical conductivity of sufficiently clean samples. Its weight D/D_0 is given by Eqn. (15) and its width $\Gamma_{\tilde{P}_{\tilde{N}_0 \tilde{M}_0}}$ by (10) and therefore the dc-conductivity should be proportional to $T^{-(2N^*-2)}$ at low T. The analysis given above was valid for very low (but finite) temperatures. For higher T, the "most relevant" approximate conservation law can heuristically be determined by the following order of magnitude estimate for the dc-conductivity

$$\sigma(\omega = 0, T) \sim \max_{N_0, M_0} \left[\frac{\chi^2_{J_x \tilde{P}_{N_0 M_0}}}{\chi_{\tilde{P}_{N_0 M_0} \tilde{P}_{N_0 M_0}}} \frac{1}{\Gamma_{\tilde{P}_{N_0 M_0}}} \right]. \tag{17}$$

Methods for a more reliable calculation are shortly discussed in the next section.

In many experimentally relevant situations, $\tilde{P}_{\tilde{N}_0 \tilde{M}_0}$ will decay by scattering from impurities. If this decay rate is sufficiently small – see Eqn. 2 – pseudo-momentum conservation will still dominate $\sigma(\omega, T)$ at low frequencies. A typical situation in a weakly disordered metal might be the following: at very low temperatures, elastic scattering with a rate Γ_{el} due to disorder determines both current and pseudo-momentum relaxation, therefore all pseudo-momenta are irrelevant and $\sigma(\omega = 0) \sim D_0/\Gamma_{el}$. The situation is more interesting at somewhat higher temperatures, where inelastic 2-particle collisions dominate, $\Gamma_{el} < \Gamma_J \propto T^2$. If $\Gamma_{el} \gg \Gamma_{\tilde{P}_{\tilde{N}_0 \tilde{M}_0}}$, then the

pseudo-momentum will decay by elastic scattering. In such a situation, we expect a well defined low-frequency peak in the optical conductivity (as shown in Fig. 1) as long as the inequality (2) is fulfilled. In such a situation, the dc-conductivity will be only weakly temperature dependent, and $\sigma(\omega = 0) \sim D(T)/\Gamma_{el}$ as the 2-particle collisions are not able to relax $\tilde{P}$. Matthiessen‘s rule is obviously not valid.

3.6. How to Calculate $\sigma(\omega, T)$ Quantitatively

As this paper focuses on qualitative arguments, we have not tried to calculate the full frequency dependence of $\sigma(\omega, T)$ for an anisotropic Fermi liquid. Here, we discuss briefly the methods which can be used to determine $\sigma(\omega)$ quantitatively.

In the language of perturbation theory, the physics discussed in this paper is the physics of vertex corrections. The presence of approximate conserved quantities implies an approximate cancellation of vertex- and self-energy corrections at low frequencies. In any perturbative calculation it is therefore essential to include the proper vertex correction. This is done automatically, if one solves the corresponding Boltzmann-type kinetic equation[2]. If, however, the pseudo-momenta are conserved by 2-particle scattering processes as discussed above, it is not sufficient to include only 2-particle processes in the collision term of the Boltzmann equation: one has to consider collisions involving N^* particles, where N^* is the smallest integer larger than $N^*_{\tilde{N}_0\tilde{M}_0}$ (8), to obtain the correct low-T behavior. Similarly, in perturbation theory, one has to include both self-energy and the corresponding vertex corrections up to order N^* in the interactions.

The fact that the pseudo-momenta decay is much slower than other degrees of freedom in the system, suggests a hydrodynamic approach to calculate $\sigma(\omega)$. This can be done with the help of the memory matrix formalism of Mori and Zwanzig[12,13]. In the context of solid state physics, the use of this method has been pioneered by Götze and Wölfle[14]. The main input of this method is the knowledge of the relevant slow variables, i.e. the pseudo-momenta. Within the memory matrix approach, the relaxation rates of those hydrodynamic variables is calculated perturbatively. The main advantage is that one can avoid to solve complicated transport- or vertex equations. The correct weights are reproduced if the relevant time scales are well seperated. For Luttinger liquids, the memory matrix has been used in Ref. 3 to calculate $\sigma(\omega)$ in situations where pseudo-momenta are approximately conserved, in Ref. 10 this method has been compared with numerical exact results for a certain classical model.

4. CONCLUSIONS

In this paper, we have discussed various approximate conservation laws which determine the low-frequency conductivity of clean anisotropic Fermi liquids with open Fermi surfaces. For a large class of situations, the dominant scattering process is ineffective and does not lead to a decay of the current due to the presence of some "protecting" pseudo-momentum which decays on a much longer time-scale. In this situation, the decay rate of the pseudo-momentum determines the dc-conductivity. The most important signature of this type of physics is a well-defined low frequency peak in the optical conductivity as is shown schematically in Fig. 1 with a weight which can be calculated from Fermi liquid theory.

This paper has investigated the transport in anisotropic Fermi liquids. The same physics is relevant not only for Luttinger liquids[3], but for example also quasi two-dimensional d-wave superconductors with nodes close to $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ as we will show in a future publication.

5. ACKNOWLEDGMENTS

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